The Optimal Allocation of Resources Among Heterogeneous Individuals

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Abstract

A planner has Atkinson-CES preference aggregation over individual outcomes and allocates $\widehat{W}$ units of discrete or bounded-continuous resources among N candidate recipients. Conditional on observables, outcomes without allocations and the marginal effects of allocations are heterogeneous across individuals. Despite combinatorial explosion with rising N, the optimal allocation function has closed-form solutions when marginal effects are non-increasing. Solutions are characterized by resource-invariant optimal allocation queues that sequence the order in which individuals begin and stop to receive allocations. The welfare distances between optimal and alternative allocations are measured in percentage resource losses as resource equivalent variations.

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1 Introduction

How should a government or an aid agency allocate limited resources among heterogeneous candidate recipients? These resources might include job training opportunities for disadvantaged workers, stimulus checks for households in a recession, or nutritional aids for young children who are at risk of undernourishment. In this paper, I provide closed-form solutions and implementation algorithms for optimal allocation problems where a planner with Atkinson-CES preferences allocates discrete or bounded-continuous units of a homogeneous resource among heterogeneous candidate recipients. The optimal solutions are determined by resource-invariant optimal allocation queues, which characterize the sequence in which individuals optimally begin and stop to receive allocations.

While solutions to optimal allocation problems are readily available when interior optima can be found via first-order conditions, in many empirical and policy-relevant settings, the optimal allocation problem can be computationally intractable. In particular, when resources come in discrete units, the size of the planning choice set combinatorially explodes as the number of candidate recipients increases.[1] Additionally, when continuous allocations are constrained by individual lower and upper bounds, the number of possibly binding cases that need to be considered in the constrained maximization problem increases exponentially as the number of candidate recipients increases.[2] Due to these computational difficulties, the allocative implications of the rich heterogeneities uncovered by structural and reduced-form empirical analysis are often not fully exploited. Researchers might only be able to consider several alternative counterfactuals rather than all constrained allocation alternatives.

I solve the problem of optimally allocating a discrete or bounded-continuous homogeneous resource among candidate recipients by first dividing the problem into an estimation-prediction space and an allocation space. The estimation-prediction space contains the model, data, and estimates from structural or reduced-form analysis. The allocation space only contains individual-

1. If N candidate recipient can each receive 0 or 1 unit of allocations from $\widehat{W}$ total units, there are N choose $\widehat{W}$ number of possible combinations. If $N = 100$ and $\widehat{W} = 50$, there are 1.0089e+29 combinations. At the other extreme, if each candidate can receive 0 to $\widehat{W}$ units of allocations, the size of the choice set becomes multinomial, and there are 1.3419e+40 combinations of possible allocation alternatives to consider for $N = 100$ and $\widehat{W} = 50$.

2. With possibly binding individual lower and upper bounds of allocation, the standard constrained maximization problem would potentially need to consider all possible binding cases where subsets of allocations are bounded below or above. Given N individual candidate recipients, the number of possible binding cases is equal to $3^N - 2^N$. When $N = 100$, there are 5.1538e+47 potential cases to consider.

specific information relevant for the allocation problem. Specifically, in the allocation space, the Atkinson-CES planner prioritizes allocations by jointly considering individual-specific *needs* and *effectiveness*. Individuals who would have the lowest levels of outcomes of interest in the absence of allocations are the most in need. Allocation increments are more effective for individuals for whom the marginal effects of incremental allocations are greater. The relative positions that a candidate recipient takes up along the optimal allocation queue are determined, in closed-form, by the planner's inequality aversion, bias, individual-specific needs, individual-specific effectiveness of incremental allocations, and individual-specific allocation bounds. When the marginal effects of allocations are non-increasing, the optimal allocation queue is invariant to aggregate resources.

The optimal allocation results can be deployed in several ways. Equipped with prior estimates and predictions, as well as observables on candidate recipients, a policymaker can use the allocation functions to determine who should receive allocations first, and how much to distribute to each. The allocation queues are functions of the policymaker's preferences as well as the constraints that the policymaker faces. An econometrician observing the allocations of the policymaker can use the optimal allocation rules to uncover planner preferences that are the most consistent with the observed choices. Given several proposed distributional schemes, a program evaluator can use the planner values along the optimal allocation queue as a scale to measure the welfare distances between policy alternatives under consideration.

Literature Review From Dixit and Stiglitz (1977) to Cunha and Heckman (2007), the assumption of CES aggregation for optimal allocation problems pervades broad areas of economics. Given what is optimal, the degree of misallocation under CES assumptions can also be evaluated (Hsieh and Klenow 2009). Results from the canonical problem, however, rely on interior solutions. I show in this paper conditions under which the CES allocation problem can be solved in closed-form when interior solutions are not possible due to discreteness in choices, or possibly binding individual lower and upper bounds on allocations. The results broaden the choice domains over which optimal allocation and misallocation problems are tractable.

A growing number of recent works in mechanism design have studied the optimal discrete choice allocation problem (Ben-Porath, Dekel, and Lipman 2014; Mylovanov and Zapechelnyuk 2017). These papers focus on allocation mechanisms that consider the trade-offs between utilitarian effectiveness and the costs of eliciting private information through costly state verification (Townsend 1979). In this paper, I allow for inequality aversion and focus on the trade-offs between needs without allocations and the effectiveness of allocations. I abstract from the information elicitation problem and solve for what is optimal conditional on the information that is currently available to the planner.

The paper relates to the empirical Rawlsian optimal targeting as well as the largely Utilitarian optimal statistical treatment literatures. There is a long line of work in development economics that studies targeting aids to the poor (Besley and Kanbur 1990; Coady, Grosh, and Hoddinott 2004; Grosh et al. 2008). Recent works have focused on using participatory wealth ranking, self-selection, and other tools beyond proxy-means tests to better identify those in need (Alatas et al. 2016; Karlan and Thuysbaert 2019). In contrast, starting with Manski (2004) and Dehejia (2005), the optimal statistical treatment literature has generally focused on allocation rules that prioritize candidate recipients with higher expected gains from treatments. Recent papers have considered binary optimal treatment rules given budget constraints, policy space constraints, and non-Utilitarian preferences (Bhattacharya and Dupas 2012; Kitagawa and Tetenov 2018, 2019). In this paper, I take existing estimates and predictions as given and solve for what is optimal for Rawlsian to Utilitarian planners given observed heterogeneities among a set of candidate recipients.

This paper also relates to a growing literature that tackles problems with combinatorially exploding choice sets (Arkolakis and Eckert 2017; Alva 2018). Empirical works have relied on algorithmic approximations of true solutions (Jia 2008; Antràs, Fort, and Tintelnot 2017; Hu and Shi 2019). In this paper, despite states-spaces that grow exponentially and choice-sets that grow factorially with the number of candidate recipients, Atkinson-CES preference aggregation and non-increasing marginal effects lead to non-backward bending resource expansion paths which can be characterized in closed-form by an optimal allocation queue. The solution falls broadly under the class of greedy algorithms: the optimal allocation queue establishes global optimality from step-wise local comparisons (Schrijver 2003).

2 The Optimal Allocation Problem

Planner preferences should allow for both inequality aversion as well as biases. A poverty alleviation campaign might focus on helping the poorest, efforts to reduce pollution might focus on lowering average emissions, and an education program might weight both average and minimum achievements. An NGO might be biased towards improving school enrollments for girls, and a development bank might be biased towards regional firms. To allow for these variations in planner preferences, given N candidate recipients of allocations indexed by i, I assume that the planner aggregates over individual-specific expected outcomes H_i with Atkinson preferences (CES aggregation) (Atkinson 1970).[3] The planner affects changes in H_i with individual-specific allocations V_i.[4] I consider the problem of maximizing the planner's objective function

$$\begin{aligned} U\left(\{H_i\}_{i=1}^{N}\right) &= \left(\sum_{i=1}^{N} \beta_i \left(H\left(V_i\,;\, \boldsymbol{x}_i, \Theta\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\,, \\ \text{where } \beta_i > 0\,,\ \sum_{i=1}^{N} \beta_i &= 1,\ \text{ and }\ -\infty < \lambda \leq 1\,, \end{aligned} \tag{1}$$

on the choice set

$$\mathcal{C} \equiv \left\{ \mathbf{V} = (V_1, \cdots, V_N) : 0 \leq V_i \in \Omega_i, \text{ and}, \sum_{i=1}^{N} V_i \leq \widehat{W} \in \mathbb{R}_+ \right\}. \tag{2}$$

Allocation V_i is constrained by aggregate resources $\widehat{W}$, as well as an individual-specific constraint set Ω_i. In empirical settings, $\boldsymbol{x}_i$ and Θ can be vectors of observables and prior estimates. $\boldsymbol{x}_i$ and Θ might also be state-space elements and calibrated parameters from a simulated model. $\boldsymbol{x}_i$ and Θ jointly determine needs, $H_i\left(V_i = 0\right)$, and effectiveness, which is $H_i\left(V_i\right) - H_i\left(V_i - 1\right)$ when allocations are discrete.

Under Atkinson-CES preference aggregation, the planner's objective can be cast as a measure of inequality. The Atkinson inequality measure is: $\left(1 - U\left(\lambda\right) \cdot \left(\sum_{i=1}^{N} H_i / N\right)^{-1}\right) \in [0,1]$, where zero indicates perfect equality. When $\lambda = 1$, the Utilitarian planner has no aversion towards inequality and maximizes the average outcome. As λ approaches $-\infty$, the planner becomes more averse to inequality and exhibits Rawlsian preferences. In between, the planner cares about inequality as well as efficiency. An important restriction is that V_i enters H_i but not H_j.

3. Atkinson-CES and equivalent preference aggregations have long been used to study allocations given inequality aversion (or equivalently, risk aversion). Other authors have studied the allocation of inputs among children by a household head (Behrman, Pollak, and Taubman 1982), districts by a city–planner (Behrman and Craig 1987) or households by a village–headman (Basurto, Dupas, and Robinson 2019). Recently, Boar and Midrigan (2020) also use Atkinson-CES aggregation to allow for inequality aversion in the analysis of optimal taxation policies.

4. H_i could be the probability of not being stunted, average school test scores, and expected utility. V_i could be food aid, additional teachers, and welfare checks. H_i, adjustable via V_i, can be inputs for CES production function U.

This means the framework here does not account for spillover effects between candidate recipients.

Given solutions to the optimal allocation problem, the expenditure minimization dual problem provides the welfare costs, in resource units, of misallocation:

Definition 1 *Given alternative allocations* $\left\{V_i^o\right\}_{i=1}^N$ *and optimal allocation* $\left\{V_i^*\left(\cdot\right)\right\}_{i=1}^N$*,*

$$\Delta^{REV} \equiv 1 - \frac{\min\left\{\widehat{W} : U\left(\left\{H_i\left(V_i^*\left(\widehat{W}\right)\right)\right\}_{i=1}^N\right) \geq U\left(\left\{H_i\left(V_i^o\right)\right\}_{i=1}^N\right)\right\}}{\Sigma_{i=1}^N V_i^o}$$

is the resource equivalent variation (REV) change.

Holding planner value constant, Δ^{REV} measures the percentage of resources that can be saved by allocating optimally. Optimal targeting could be costly administratively and difficult politically (Coady, Grosh, and Hoddinott 2004). Δ^{REV} provides a common scale in resource units to measure the welfare distances among alternative allocations. Alternative allocation rules might be random, uniform, optimally chosen based on limited or augmented candidate recipient observables $\boldsymbol{x}_i$, or optimally chosen based on tightened or relaxed allocation constraints Ω_i.

2.1 Deficiencies of the CES Allocation Problem

When H_i is proportional to V_i and $\alpha_i > 0$, the planner solves the workhorse CES problem:

$$\max_{\{V_i\}_{i=1}^N} \left(\sum_{i=1}^N \beta_i \left(H_i\right)^\lambda\right)^{\frac{1}{\lambda}} \tag{3}$$
$$\text{s.t.} \,\forall\, i,\;\; H_i = \alpha_i V_i \;\text{ and }\; 0 \leq V_i\,,\text{ and } \Sigma_{i=1}^N V_i = \widehat{W}$$

Analytical solutions are immediate from First Order Conditions. Allowing for $\alpha_i \neq \alpha_j$ nests individual-specific CES functions with constant returns. If H_i can be reallocated with homogeneous marginal effects ($H_i = \alpha V_i \in \mathbb{R}_+$) and $\beta_i = \frac{1}{N}$, it would be optimal to equalize V_i for all $\lambda \in (-\infty, 1]$ (Atkinson 1970).

The CES problem makes three key allocative assumptions related to aspects of the Inada conditions. First, under CES, $H_i\left(V_i = 0\right) = 0$. However, most allocation schemes provide supplemental resources to enhance outcomes, which means $H_i\left(V_i = 0\right) > 0$. Second, under CES, the objective function is continuously differentiable in V_i. Empirically, resources often have to be provisioned in discrete units. Third, except for the aggregate resource constraint, the

CES problem does not impose possibly binding individual constraints on V_i. In practice, there are often both minimum requirements as well as upper limits on allocations.[5] When these CES assumptions are violated, the objective function would no longer be homothetic in allocations, and the elasticity of substitution for allocations would not be constant.

In the following sections, preserving Atkinson-CES preference aggregation, I provide solutions to the discrete and bounded-continuous allocation problems.

3 Discrete Allocation Space

3.1 The Discrete Allocation Problem

A food program needs to determine how many bags of rice to provide to each household in need. A maternal health program needs to determine the number of pre-natal check-up slots to provide to each mother. A job training program needs to assign finite training spots to heterogeneous unemployed individuals. The discrete constraint set models these choice as

$$\mathcal{C}^D \equiv \left\{ \mathbf{D} = (D_1, \cdots, D_N) : D_i \in \left\{ \underline{D}_i, \underline{D}_i + 1, \cdots, \bar{D}_i \right\}, \underline{D}_i \in \mathbb{N}_0, \sum_{i=1}^{N} D_i \leq \widehat{W} \right\}. \quad (4)$$

$\mathcal{C}^D$ allows for individual-specific upper and lower bounds on allocations D_i. A special case is the binary choice set, where $\underline{D}_i = \underline{D} = 0$ and $\bar{D}_i = \bar{D} = 1$. More generally, the discrete constraint set can be used to approximate continuous choice sets. In standard discrete choice problems, the optimal choice is found by comparing utility at all possible choices (Train 2009). The size of $\mathcal{C}^D$, however, grows factorially with N and $\bar{D}_i$. Evaluating utility at all combinatorial possibilities quickly becomes computationally infeasible.[6]

For individual i, l indexes each increment of discrete allocations. Let α_{il} denote the

5. For example, suppose H_i equals one minus the probability of undernourishment and V_i is food aid. Without food aid, the chance of undernourishment would not be one. Food aid might arrive in the form of bags of rice or bottles of nutritional supplements, and the amount of food that an individual can consume is finite.

6. A number of recent works have studied combinatorial optimization in economic settings (Jia 2008; Arkolakis and Eckert 2017). In general, exact solutions are not possible. Here, at the lower-end, where $\bar{D}_i = 1$, the number of allocation possibilities is binomial, $N!/\left(\left(N - \widehat{W}\right)!\widehat{W}!\right)$. At the upper-end, where $\bar{D}_i = \widehat{W}$, the choice set is multinomial, $\left(\widehat{W} + N - 1\right)!/\left((N-1)!\widehat{W}!\right)$. In the language of shuffle algebra, the size of the choice set is equal to all combinations of the first $\widehat{W}$ cards among all interwoven shuffles of N card decks with $\bar{D}_i$ cards per deck.

effectiveness of allocations and A_i denote *needs*:

$$H_i^D = A_i + \sum_{l=1}^{\bar{D}_i} \left(\alpha_{il} \cdot \mathbb{1}\{l \leq D_i\} \right) . \tag{5}$$

To gain tractability, I impose three restrictions on α_{il} and A_i:

Assumption 1 *Marginal effects α_{il} for the l^{th} increment of D_i on H_i are: (1) positive, $\alpha_{il} > 0$; (2) non-increasing, $\alpha_{il} \leq \alpha_{i,l-1}$; and (3) can lead to positive outcomes, $A_i + \sum_{l=1}^{\bar{D}_i - 1} \alpha_{il} > 0$.*

Assumption 1 imposes no functional form or parametric assumptions on the underlying reduced form or structural model. The first restriction is innocuous. If additional increments have zero effects, $\bar{D}_i$ can be lowered. The second restriction accommodates both constant returns as well as arbitrary step functions of decreasing returns.[7] The third restriction allows for $A_i > 0$. Additionally, $A_i < 0$ is also possible if the cumulative effects of allocations lead to positive outcomes (Geary 1950).[8] Given these, without loss of generality, the minimum allocation can be subsumed under A_i and $\underline{D}_i$ set to zero.

A_i and α_{il} are functions of $(\boldsymbol{x}_i, \Theta)$. The parameters could capture the reduced-form effects of stimulus checks on consumption. For notational simplicity, I assume that the check is of a fixed amount and the allocation problem is binary. H_i can be the expected average consumption: $H_i(x_i, D_i) = \theta_0 + \theta_1 x_i + \theta_2 D_i - \theta_3 D_i \mathbb{1}\{x_i \geq q\}$. Then, $A_i = \theta_0 + \theta_1 x_i$ is the expected consumption as a function of age x_i without allocations. $\alpha_i = \theta_2$ is the expected consumption increase from the stimulus check for individuals younger than age q, and $\alpha_i = \theta_2 - \theta_3 > 0$ is the potentially smaller consumption effects of stimulus on older individuals who have higher levels of savings. When the distribution of uncertainty is available, the H_i for a planner with individual-specific inequality aversion can also be: $H_i(x_i, D_i) = \left(\int (\theta_0 + \theta_1 x_i + \theta_2 D_i - \theta_3 D_i \mathbb{1}\{x_i \geq q\} + \epsilon)^{\sigma_i} f(\epsilon) d\epsilon \right)^{\frac{1}{\sigma_i}}$. In this case, $A_i(x_i) = \left(\int (\theta_0 + \theta_1 x_i + \epsilon)^{\sigma_i} f(\epsilon) d\epsilon \right)^{\frac{1}{\sigma_i}}$, and $\alpha_i(x_i) = H_i(x_i, D_i = 1) - A_i(x_i)$.

A_i and α_{il} could also incorporate predictions from structural models. One might have $H(S^*(D_i, \boldsymbol{x}_i, \Theta), \boldsymbol{x}_i, \Theta)$. Consumption H_i is a function of the savings policy function S^*, which is endogenous to the provision of stimulus check D_i. Nygaard, Sørensen, and Wang (2020)

7. In some settings, there might be threshold levels of allocations where returns shift from increasing to non-increasing. Assumption 1 remains valid if the planner sets minimum allocation level $\underline{D}_i$ at these thresholds.

8. In restriction 3, the summation ends at $\bar{D}_i - 1$ so that not allocating up to the maximum remains a valid choice.

solve an optimal stimulus check allocation problem in the context of a dynamic life-cycle model using the algorithms provided in this paper. The authors compute α_{il} over 244 stimulus check increments (at \$100 per increment) for a variety of household types.

A_i and α_{il} transform the problem from the estimation-prediction space to the dimension-reduced allocation space. I define the solution to the discrete optimal allocation problem as:

Definition 2 *Given N individuals, the solutions to the discrete optimal allocation problem are allocation functions* $D_j^*\left(\widehat{W}, \lambda, \{\beta_i\}_{i=1}^N, \{A_i\}_{i=1}^N, \left\{\{\alpha_{il}\}_{l=1}^{\bar{D}_i}\right\}_{i=1}^N, \{\underline{D}_i, \bar{D}_i\}_{i=1}^N\right) : \mathbb{N} \times (-\infty, 1] \times (0,1)^N \times \mathbb{R}^N \times \mathbb{R}_+^{\left(\sum_{i=1}^N \bar{D}_i\right)} \times \mathbb{N}_0^{(N\cdot 2)} \to \left\{\underline{D}_j, \underline{D}_j + 1, \cdots, \bar{D}_j\right\}$ *such that* $\mathbf{D}^* = (D_1^*, \cdots, D_N^*)$ *maximizes,*

$$\max_{\mathbf{D}\in\mathcal{C}^D} \left(\sum_{i=1}^N \beta_i \left(A_i + \sum_{l=1}^{\bar{D}_i} \left(\alpha_{il} \cdot \mathbb{1}\{l \leq D_i\}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}, \tag{6}$$

on the constraint set $\mathcal{C}^D\left(\widehat{W}, \{\underline{D}_i, \bar{D}_i\}_{i=1}^N\right)$.

3.2 Discrete Allocation Solutions

The solution concept here is to solve for an optimal allocation queue that is invariant to resources. Given N individuals, the queue is ranked from 1 to $\sum_{i=1}^N \bar{D}_i$, where 1 indicates the top-ranked individual who would receive the first unit of allocation. Suppose an individual has two potential units of allocations ranked at the 1st and the 4th spot of the queue; if $\widehat{W} = 4$, the individual receives both units of allocation. Under Assumption 1, as aggregate resources increase, the planner will only allocate more to individuals—the discrete resource (income) expansion path does not bend backward. The optimal allocation queue characterizes the resource expansion path.

Theorem 1 *Suppose that Assumption 1 holds and assume without loss of generality* $\underline{D}_i = 0$ *for all i. The Atkinson-CES planner's discrete allocation solutions,* $D_1^*, \cdots, D_N^*$, *are:*

$$D_i^*\left(\widehat{W}\right) = \sum_{l=1}^{\bar{D}_i} \mathbb{1}\left\{Q_{il}^D \leq \widehat{W}\right\} \tag{7}$$

where Q_{il}^D is the position of the l^{th} allocation increment for individual i on the allocation queue,

$$Q_{il}^D = \sum_{\widetilde{i}=1}^{N} \sum_{\widetilde{l}=1}^{\bar{D}_{\widetilde{i}}} \mathbb{1}\left\{ \frac{\beta_{\widetilde{i}}}{\beta_i} \cdot \left(\frac{\left(A_{\widetilde{i}} + \sum_{l'=1}^{\widetilde{l}} \alpha_{\widetilde{i}l'}\right)^{\lambda} - \left(A_{\widetilde{i}} + \sum_{l'=1}^{\widetilde{l}-1} \alpha_{\widetilde{i}l'}\right)^{\lambda}}{\left(A_i + \sum_{l'=1}^{l} \alpha_{il'}\right)^{\lambda} - \left(A_i + \sum_{l'=1}^{l-1} \alpha_{il'}\right)^{\lambda}} \right) \geq 1 \right\} . \tag{8}$$

Additionally, given alternative allocations D_i^o and $\widehat{W}^o = \sum_{i=1}^N D_i^o$, Δ^{REV} is:

$$\widehat{W}^o \cdot \left(1 - \Delta^{REV}\right) = \min \left\{ \widehat{W} : \frac{\sum_{i=1}^N \beta_i \cdot \left(A_i + \sum_{l=1}^{\bar{D}_i} \left(\alpha_{il} \cdot \mathbb{1}\{Q_{il}^D \leq \widehat{W}\}\right)\right)^{\lambda}}{\sum_{i=1}^N \beta_i \cdot \left(A_i + \sum_{l=1}^{D_i^o} \alpha_{il}\right)^{\lambda}} \geq 1 \right\} \tag{9}$$

The proof for Theorem 1 is provided in the Appendix.

There are four key aspects to Theorem 1. First, the ranking of allocation increments along the optimal allocation queue, Q_{il}^D, is based on a comparison of the level of individual outcomes with and without the next increment of allocation, scaled by inequality aversion λ, and weighted by individual bias β_i. Specifically, for the l^{th} allocation increment for individual i, only $\beta_i \left(\left(A_i + \sum_{l'=1}^{l} \alpha_{il'}\right)^{\lambda} - \left(A_i + \sum_{l'=1}^{l-1} \alpha_{il'}\right)^{\lambda} \right)$ needs to be evaluated. In the boundary case where $\lambda = 1$ and $\beta_i = \frac{1}{N}$, Equation (8) simplifies to a descending sort over α_{il}. As inequality aversion shifts away from $\lambda = 1$, the importance of A_i increases. In the case of binary allocations, at $\lambda = -\infty$, the allocation queue simplifies to an ascending sort over A_i, where individuals with the lowest needs receive allocations first.

Second, the optimal allocation queue is invariant to resources—$\widehat{W}$ does not appear in Equation (8). Aggregate resources only act as cut-off points in Equation (7). The optimal allocation queue provides the solution to the optimal allocation problem from when $\widehat{W} = 0$ to $\widehat{W} = \sum_i^N \bar{D}_i$. The ranking for the same individual, however, can change when a program expands to consider additional candidate recipients, when additional observables become available, or when a program changes its service mandates.

Third, the Δ^{REV} welfare comparison statistics is also a function of the optimal allocation queue. The welfare value of any alternative allocations can be readily mapped to a position along the queue. Δ^{REV} is then directly measurable as the distance between positions along the optimal queue.

Fourth, Theorem 1 accommodates ties and group-aggregation. When the number of candidate recipients of the same type increases, the relative rankings across types are preserved. When policy constraints limit the covariates that could be used to condition allocations (Kitagawa and Tetenov 2018), A_i and α_{il} can be redefined with group-specific aggregation functions.[9]

Given inputs α_{il} and A_i, the computational implementation of Theorem 1 in the companion code package solves for optimal allocation queues along with welfare along the queues given a vector of inequality aversion λ. Since the computational burden increases linearly with N, problems with combinatorially exploding choice sets can be readily solved even when N is large. The main computational burden of implementing Theorem 1 potentially falls on evaluating $\alpha_{il}(\boldsymbol{x}_i, \Theta)$ based on model predictions for many units of allocation increments.

4 Bounded-Continuous Allocation Space

4.1 The Bounded-Continuous Allocation Problem

A poverty alleviation program might provide continuous transfers to households. An early-childhood program might provide protein supplements in grams to children-at-risk. Income security needs, dietary energy requirements, or pre-existing allocations might lead to lower bounds on allocations. Physical limits on intakes, policy constraints, or limits on prediction extrapolations can lead to upper bounds on allocations. The bounded-continuous choice set is:

$$\mathcal{C}^C \equiv \left\{ \mathbf{C} = (C_1, \cdots, C_N) : 0 \leq \underset{\sim}{C}_i \leq C_i \leq \bar{C}_i, \text{ and, } \sum_{i=1}^{N} C_i \leq \widehat{W} \right\}. \tag{10}$$

Given $\mathcal{C}^C$, solutions to the planner's problem need to satisfy Karush-Kuhn-Tucker conditions. With complementary slackness, standard solution methods require a comparison of values across all binding possibilities.[10] The number of possible binding cases grows exponentially with N and quickly becomes computationally intractable. Allocation for each individual might be

9. Suppose an Atkinson-CES planner observes education, age, and associated model predictions, but can only allocate based on education. Then, A_i can be a CES aggregate of group-member outcomes when allocations are zero given the conditional distribution of age on education. α_{il} is similarly aggregated with group members receiving the same incremental levels of allocations. Relative positions along the optimal queue are determined by education groups. Individuals are tied within group. In the context of an optimal stimulus check problem, Nygaard, Sørensen, and Wang (2020) provide examples for including and excluding age as a policy conditioning variable.

10. Bounded-continuous problems appear, for example, in models with possibly binding financial constraints. Tackling dynamics, solutions methods are generally algorithmic (Guerrieri and Iacoviello 2015).

separately bounded below, bounded above, or unbounded. The maximum potential number of cases to consider is $3^N - 2^N$, which is the total number of edges and faces of an N^{th} dimensional hypercube.

Relaxing the CES proportionality assumption, I allow expected outcomes to be linear in C_i,

$$H_i^D = A_i + \alpha_i \cdot C_i \ . \tag{11}$$

For closed-form tractability, Equation (11) assumes that C_i has constant-returns. Overall, I impose three restrictions on α_i (effectiveness) and A_i (needs):

Assumption 2 *Marginal effects α_i of C_i on expected outcome H_i are: (1) positive, $\alpha_i > 0$; (2) constant over C_i increments; and (3) can lead to positive expected outcomes, $A_i + \alpha_i \cdot \bar{C}_i > 0$.*

Assumption 2 allows for $A_i \neq 0$, which permits subsuming minimum allocations under A_i. I henceforth assume for notational clarity that $\underline{C}_i = 0$ for all i. In the case of linear regressions, A_i could be the dot product of covariate estimates and covariate observables, and α_i could capture treatment effects along observable dimensions of heterogeneity. In structural settings, C_i can be transfers to individuals who solve individual-specific constant-returns utility maximization problems over individual inputs. In such contexts, α_i is a function of individual-specific prices, elasticities, and productivities, and A_i captures individual characteristics that determine initial conditions.[11]

The solution to the constant-returns bounded-continuous allocation problem is defined as:

Definition 3 *The solutions to the constant-returns bounded-continuous optimal allocation problem are allocation functions $C_j^*\left(\widehat{W}, \lambda, \{\beta_i\}_{i=1}^N, \{A_i\}_{i=1}^N, \{\alpha_i\}_{i=1}^N, \{\underline{C}_i, \bar{C}_i\}_{i=1}^N\right) : \mathbb{R}_+ \times (-\infty, 1] \times (0,1)^N \times \mathbb{R}^N \times \mathbb{R}_+^N \times \mathbb{R}_+^{(N \cdot 2)} \rightarrow \left[\underline{C}_j, \bar{C}_j\right]$ such that $\mathbf{C}^* = (C_1^*, \cdots, C_N^*)$ maximizes,*

$$\max_{C \in \mathcal{C}^C} \left(\sum_{i=1}^{N} \beta_i \left(A_i + \alpha_i \cdot C_i\right)^{\lambda} \right)^{\frac{1}{\lambda}} \ , \tag{12}$$

on the constraint set $\mathcal{C}^C\left(\widehat{W}, \{\underline{C}_i, \bar{C}_i\}_{i=1}^N\right)$.

11. With one individual-specific input, α_i is the ratio of marginal effects and price. Generally, constant-returns models with a separable term for initial conditions could be accommodated. This includes, for example, the canonical early and late childhood investment model of Cunha and Heckman (2007).

It should be noted that while Theorem 1 can approximate the solutions to bounded-continuous problems, solving the problem in Equation (12) offers two advantages when the marginal effects of incremental allocations are approximately linear. First, the solutions of Equation (12) is a function of $\{\alpha_i\}_{i=1}^{N}$. This avoids the need to evaluate increment-specific effects of allocations. Second, the bounded-continuous problem accommodates heterogeneous prices through α_i.[12]

4.2 Bounded-Continuous Allocation Solutions

Following the solutions to the discrete problem, to derive optimal policy functions, I first order candidate recipients along four allocation sequences:

$$\begin{aligned} &\text{start-queue } \underline{\mathbf{Q}}^C = \left\{\underline{Q}_1^C, \cdots, \underline{Q}_N^C\right\}, \text{ stop-queue } \overline{\mathbf{Q}}^C = \left\{\bar{Q}_1^C, \cdots, \bar{Q}_N^C\right\}, \\ &\text{start-knots } \underline{\mathbf{K}}^C = \left\{\underline{K}_1^C, \cdots, \underline{K}_N^C\right\}, \text{ and stop-knots } \overline{\mathbf{K}}^C = \left\{\bar{K}_1^C, \cdots, \bar{K}_N^C\right\}. \end{aligned} \tag{13}$$

Queues $\underline{\mathbf{Q}}^C$ and $\overline{\mathbf{Q}}^C$ index the order in which individuals begin or stop to receive additional allocations. Knots $\underline{\mathbf{K}}^C$ and $\overline{\mathbf{K}}^C$ mark the aggregate resource levels where individuals join and leave the queues. Intuitively, marginal optimality conditions apply—successively along the queue—relative to the first allocation recipient $i = \mathbb{I}$, for whom $\underline{Q}_{i=\mathbb{I}}^C = 1$ and $\underline{K}_{i=\mathbb{I}}^C = 0$.

Values for elements of the four sequences are derived from First Order Conditions. For all $i, j \in (1, \cdots, N)$, when individual allocation bounds do not bind, relative optimality requires that:

$$C_i^{rela}\left(C_j\right) = \underbrace{\left(\left(\frac{\beta_i \alpha_i}{\beta_j \alpha_j}\right)^{\frac{1}{1-\lambda}} \frac{A_j}{\alpha_i} - \frac{A_i}{\alpha_i}\right)}_{\phi_{i,j}^{\mathrm{y}}} + \underbrace{\left(\left(\frac{\beta_i}{\beta_j}\right)^{\frac{1}{1-\lambda}} \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{\lambda}{1-\lambda}}\right)}_{\phi_{i,j}^{\mathrm{s}}} \cdot C_j, \tag{14}$$

with relative y-intercept $\phi_{i,j}^{\mathrm{y}}$, slope $\phi_{i,j}^{\mathrm{s}}$, and x-intercept $\phi_{i,j}^{\mathrm{x}} = -\phi_{i,j}^{\mathrm{y}}/\phi_{i,j}^{\mathrm{s}}$.

To arrive at the four sequences, first, $\underline{Q}_i^C$ is ordered by the sign of $\phi_{i,j}^{\mathrm{y}}$.[13] Second, $\underline{K}_i^C$ is the sum of all truncated relative optimal choices $\left\{ \min\left(\bar{C}_j, \max\left(0, C_j^{rela}\left(C_{\mathbb{I}}\right)\right)\right) \right\}_{j=1}^{N}$ with respect to the first recipient at $C_{\mathbb{I}} = \phi_{i,\mathbb{I}}^{\mathrm{x}}$. Third, $\bar{Q}_i^C$ is determined by upper-bound-intercepts—

12. Heterogeneous prices might arise when a food aid agency distributes money to localities where prices for the same nutritional inputs differ. Due to indivisibility, the presence of heterogeneous prices can lead to backward bending resource expansion paths for discrete problems. See Appendix A.2 for details.

13. If allocation $C_j = \phi_{i,j}^{\mathrm{x}}$, relative optimality requires that $C_i = 0$. If allocation C_j exceeds $\phi_{i,j}^{\mathrm{x}}$, relative optimality requires that $C_i > 0$. Hence, with respect to j, if $\phi_{i,j}^{\mathrm{x}} > 0$—which means $\phi_{i,j}^{\mathrm{y}} < 0$—then i should be ranked after j on the optimal start-queue.

the order in which C_i reaches its upper bound $\bar{C}_i$. Fourth, $\bar{K}_i^C$ is the sum of bounded allocations at these upper-bound-intercepts.

Theorem 2 *Suppose that Assumption 2 holds and assume without loss of generality $\underline{C}_i = 0$ for all i. The Atkinson-CES planner's constant-returns bounded-continuous allocation solutions, $C_1^*, \cdots, C_N^*$, are:*

$$C_i^*\left(\widehat{W}\right) = \min\left\{\bar{C}_i,\ \max\left\{0,\ \phi_{i,\mathbb{I}}^y + \phi_{i,\mathbb{I}}^s \cdot C_{\mathbb{I}}\left(\widehat{W}\right)\right\}\right\}, \tag{15}$$

$$C_{\mathbb{I}}\left(\widehat{W}\right) = \frac{\widehat{W} - \sum_{j=1}^N \left(\phi_{j,\mathbb{I}}^y \cdot \mathbb{1}\left\{\underline{K}_j^C < \widehat{W} \leq \bar{K}_j^C\right\} + \bar{C}_j \cdot \mathbb{1}\left\{\bar{K}_j^C < \widehat{W}\right\}\right)}{\sum_{j=1}^N \left(\phi_{j,\mathbb{I}}^s \cdot \mathbb{1}\left\{\underline{K}_j^C < \widehat{W} \leq \bar{K}_j^C\right\}\right)}. \tag{16}$$

Start-queue $\underline{Q}_i^C$, stop-queue $\bar{Q}_i^C$, start-knots $\underline{K}_i^C$ and stop-knots $\bar{K}_i^C$ values for individual i are:

1. $\mathbb{I} = \arg\min_{i \in \{1,\ldots,N\}} \left\{\underline{Q}_i^C\right\},\ \underline{Q}_i^C = \sum_{j=1}^N \mathbb{1}\left\{\left(\frac{A_j}{A_i}\right) \cdot \left(\frac{\alpha_j \cdot \beta_j}{\alpha_i \cdot \beta_i}\right)^{\frac{1}{\lambda-1}} \leq 1\right\}$
2. $\underline{K}_i^C = \sum_{j=1}^N \left(\min\left\{\bar{C}_j, \left(\phi_{j,\mathbb{I}}^y + \phi_{j,\mathbb{I}}^s \cdot \frac{-\phi_{i,\mathbb{I}}^y}{\phi_{i,\mathbb{I}}^s}\right)\right\} \cdot \mathbb{1}\left\{\underline{Q}_j^C \leq \underline{Q}_i^C\right\}\right)$
3. $\bar{Q}_i^C = \sum_{j=1}^N \mathbb{1}\left\{\left(\frac{\bar{C}_i - \phi_{i,\mathbb{I}}^y}{\phi_{i,\mathbb{I}}^s}\right) \cdot \left(\frac{\phi_{j,\mathbb{I}}^s}{\bar{C}_j - \phi_{j,\mathbb{I}}^y}\right) \geq 1\right\}$
4. $\bar{K}_i^C = \sum_{j=1}^N \left(\bar{C}_j \cdot \mathbb{1}\left\{\bar{Q}_j^C \leq \bar{Q}_i^C\right\} + \max\left\{0, \left(\phi_{j,\mathbb{I}}^y + \phi_{j,\mathbb{I}}^s \cdot \frac{\bar{C}_i - \phi_{i,\mathbb{I}}^y}{\phi_{i,\mathbb{I}}^s}\right)\right\} \cdot \mathbb{1}\left\{\bar{Q}_j^C > \bar{Q}_i^C\right\}\right)$.

Additionally, given alternative allocations C_i^o and $\widehat{W}^o = \sum_{i=1}^N C_i^o$, Δ^{REV} is:

$$\widehat{W}^o\left(1 - \Delta^{REV}\right) = \min\left\{\widehat{W} : \frac{\sum_{i=1}^N \beta_i \left(A_i + \alpha_i \cdot C_i^*\left(\widehat{W}\right)\right)^\lambda}{\sum_{i=1}^N \beta_i \left(A_i + \alpha_i \cdot C_i^o\right)^\lambda} \geq 1\right\}. \tag{17}$$

The proof for Theorem 2 is illustrated in the Appendix.

A core result of Theorem 2 is that, like Theorem 1, the allocation queues are invariant to aggregate resources $\widehat{W}$. In particular, $\underline{Q}_i^C$, $\bar{Q}_i^C$, $\underline{K}_i^C$, and $\bar{K}_i^C$ are not functions of $\widehat{W}$. This means that the optimal allocation problem can be solved at once to provide solutions along all resource levels from $\widehat{W} = 0$ to $\widehat{W} = \sum_{i=1}^N \bar{C}_i$.

Theorem 2 also shows that candidate recipients along allocation queues are sorted by simple functions of A_i, α_i, β_i, and $\bar{C}_i$. In particular, rankings along the start queue is determined by relative values of $A_i / (\alpha_i\beta_i)^{\frac{1}{1-\lambda}}$, where λ determines the relative importance of A_i and α_i. At the extremes, the start queue of the Utilitarian planner is a descending sort over $\alpha_i\beta_i$, and the start queue of the Rawlsian planner is an ascending sort of A_i. In the absence of upper bounds on allocations and assuming $\beta_i = \frac{1}{N}$, the Utilitarian planner allocates all resources to the individual with the highest α_i. In contrast, in the absence of upper bounds, the Rawlsian planner equalizes the outcome of interest as much as possible by successively adding higher A_i individuals to the allocation queue.[14]

The solution to the bounded-continuous problem includes many corners. The closed-form solution presented in Theorem 2 is possible despite the lack of interior solutions because after identifying the first recipient of allocations along the optimal start queue, the allocation problem for N candidate recipients is restated in Equation (16) as an allocation problem for the first recipient along the allocation queue. Given the constant-returns assumption, Equation (16) is simply a linear spline. Optimal allocations for all other candidate allocation recipients are then directly obtainable through First Order Conditions given bounds, as shown in Equation (15).

The algorithmic implementation of Theorem 2 is provided in the companion programs. First, the four allocation sequences—$\underline{\mathbf{Q}}^C$, $\overline{\mathbf{Q}}^C$, $\underline{\mathbf{K}}^C$ and $\overline{\mathbf{K}}^C$—are computed. Second, given resource availability, the allocation for the first recipient along the start queue is found. Third, allocations for all other recipients are computed. The computational burden rises linearly with N and involves evaluating $N \cdot 2$ knots of a linear spline. In comparison to the discrete allocation problem, the computational burden in the estimation-prediction step is lower since only one marginal effect per candidate recipient needs to be computed.

5 Examples

5.1 Discrete Allocation Example $N = 2$

I solve an illustrative discrete allocation problem based on Equation (6) when $N = 2$. Figure 1 presents the results and shows expected outcomes for individuals $i = 1, 2$ along the axes. Parameters for both individuals conform to Assumption 1.[15] Visually, the feasible expected

14. When $\bar{C}_i = \infty$ for all i, the Rawlsian planner pours water down a two-dimensional stairwell: the level of each step is A_i, the width of each step is α_i, flat water level reaches higher steps as $\widehat{W}$ increases, and $\widehat{W}$ equals total water area.

15. A_i is positive, α_{il} is positive and strictly decreasing, $\underline{D}_i$ is equal to zero, and I let $\bar{D}_{i=1} = 6$ and $\bar{D}_{i=2} = 5$.

outcome set forms a rectilinear grid with diminishing distances towards the top-right.[16]

I let $A_1 > A_2$ and $\alpha_{1,l} > \alpha_{2,l}$ for all l, and solve for optimal allocations at possible $\widehat{W}$ values under unbiased planners ($\beta_i = 0$). With greater needs and lower effectiveness, individual $i = 2$ receives more allocations from the equality-centric planner ($\lambda = -100$). In contrast, individual $i = 1$ receives more from the efficiency-focused planner ($\lambda = 0.99$). Allocations for the approximately Cobb-Douglas planner ($\lambda = -0.01$) lies between the two boundary planners. Given non-increasing and positive α_{il} and Atkison-CES preferences, the discrete resource expansion paths do not bend backward. Theorem 1 traces out the positions that individual allocations take along each resource expansion path. As N increases, Theorem 1 provides a tractable way to jointly consider equity and efficiency given heterogeneous needs and effectiveness.

Theorem 1 can also be used in mapping observed choices to preferences. Suppose observed allocations are $D_1^o = D_2^o = 3$. In Figure 1, $D_1^* = D_2^* = 3$ for both $\lambda = -100$ and $\lambda = -0.01$, but not for $\lambda = 0.99$. Conditional on $\beta_{i=1} \in (0, 1)$, one could solve for the range of λ that rationalizes $\left\{D_i^o\right\}_{i=1}^2$. As N increases, conditional on λ, joint bounds on $\{\beta_i\}_{i=1}^N$ could be constructed that rationalize $\left\{D_i^o\right\}_{i=1}^N$. Theorem 1, by providing preference-specific analytical allocative solutions, allows calibration and estimation procedures to search through large parameter spaces at low costs.

5.2 Bounded-Continuous Example $N = 2$

I consider in this section the bounded-continuous allocation problem in Equation (12) when $N = 2$. Figure 2 illustrates the results. I assume that $\alpha_i > 0$ and $A_i > 0$ for $i = 1, 2$. The maximum feasible expected outcome rectangle has left bottom vertex (A_1, A_2), and top right vertex $(A_1 + \alpha_1 \bar{C}_1, A_2 + \alpha_2 \bar{C}_2)$. Following the discrete $N = 2$ example, I assume that $A_1 > A_2$ and $\alpha_1 > \alpha_2$. As in the discrete problem, the resource expansion paths do not bend backward in Figure 2.

Figure 2 visualizes the thresholds where individuals are introduced or dropped from receiving additional allocations. For example, for $\lambda = -0.01$, individual $i = 1$ starts receiving first and ends receiving last: $\left(\underset{\sim}{Q}_1^C = 1, \bar{Q}_1^C = 2, \underset{\sim}{Q}_2^C = 2, \bar{Q}_2^C = 1\right)$. Given these, following Equation

16. Expected outcomes form a grid because D_i only impacts H_i, such that the effect of additional $D_{i=1}$ on $H_{i=1}$ is not dependent on the value for $D_{i=2}$. Given that $\alpha_{il} > 0$, each point on the expected outcome grid corresponds to a point on the choice grid. And since $\alpha_{il} > \alpha_{i,l+1}$, the rectangular areas between expected outcome vertices are decreasing towards the top-right of Figure 1, forming a rectilinear grid.

(16) from Theorem 2, we have:

$$
\begin{aligned}
C_1\left(\widehat{W}\right) = \widehat{W} \cdot \mathbb{1}\left\{\widehat{W} \leq \phi^{\mathrm{x}}_{2,1}\right\} \\
+ \left(\frac{\widehat{W} - \phi^{\mathrm{x}}_{2,1}}{1 + \phi^{\mathrm{s}}_{2,1}} + \phi^{\mathrm{x}}_{2,1}\right) \cdot \mathbb{1}\left\{\phi^{\mathrm{x}}_{2,1} < \widehat{W} \leq \left(\bar{C}_2 + \frac{\bar{C}_2 - \phi^{\mathrm{y}}_{2,1}}{\phi^{\mathrm{s}}_{2,1}}\right)\right\} \\
+ \left(\widehat{W} - \bar{C}_2\right) \cdot \mathbb{1}\left\{\left(\bar{C}_2 + \frac{\bar{C}_2 - \phi^{\mathrm{y}}_{2,1}}{\phi^{\mathrm{s}}_{2,1}}\right) < \widehat{W} \leq \left(\bar{C}_1 + \bar{C}_2\right)\right\} .
\end{aligned}
\tag{18}
$$

The optimal allocation spline has three linear segments with four associated knots:

$$
\begin{aligned}
\underline{K}^C_1 = 0\,,\ \underline{K}^C_2 = \phi^{\mathrm{x}}_{2,1}\,, \\
\bar{K}^C_1 = \bar{C}_1 + \bar{C}_2\,,\ \bar{K}^C_2 = \left(\bar{C}_2 + \frac{\bar{C}_2 - \phi^{\mathrm{y}}_{2,1}}{\phi^{\mathrm{s}}_{2,1}}\right).
\end{aligned}
\tag{19}
$$

Along the middle segment, $i = 1$ receives $\frac{1}{1+\phi^{\mathrm{s}}_{2,1}}$ fraction of each unit of additional resources.

Theorem 2 provides a closed-form accounting of all $N \cdot 2$ entry and exit points along each resource expansion path. Visually, as N increases, Figure 2 gains exponentially more possibly binding edges and faces. However, the linear spline from Equation (16) remains one dimensional for all N. The introduction of an additional candidate recipient only increases the total number of knots by two.

Theorem 2 also facilitates welfare comparisons. Aggregate welfare at alternative allocations C^{o}_1 and C^{o}_2 must be no better than welfare at $C^*_1\left(\widehat{W} = C^{\mathrm{o}}_1 + C^{\mathrm{o}}_2\right)$ and $C^*_2\left(\widehat{W} = C^{\mathrm{o}}_1 + C^{\mathrm{o}}_2\right)$. The expenditure minimization problem from Equation (17) traces backward along each resource expansion path in Figure 2 until the value given resources falls below the utility at alternative allocations. Closed-form solutions allow for immediate welfare evaluations along all paths and the relative welfare comparisons of optimal and alternative allocations.

5.3 Empirical Examples

The allocation algorithms could be applied in various structural and reduced-form empirical settings. Nygaard, Sørensen, and Wang (2020) use the allocation algorithms to determine the optimal allocation of stimulus checks among United States households that differ in income, marital status, and the number of children. For these heterogeneous households, the authors use a calibrated dynamic life-cycle consumption-saving model to predict the level of consumption in the absence of allocations as well as the marginal effects of incremental stimulus checks on consumption. Given varying constraints on the minimum and maximum allocation bounds for different types of households, the Δ^{REV} gaps between constrained optimal allocations and the actual allocations from the United States Coronavirus Aid, Relief, and Economic Security (CARES) Act of March 2020 are computed.

In this section, I provide a reduced-form implementation example of the allocation algorithms. To illustrate ideas, I take the concept of the optimal allocation queue to the National Supported Work Demonstration (NSW) dataset from LaLonde (1986). The dataset includes 297 adult males who were treated in a job training program and 425 adult males who were in the control group. LaLonde (1986) reports a significant gain of \$886 in 1979 wage for the treatment group.

I estimate the effects of job training (binary) on employment probability using a logistic regression model. A_i is the expected probability of employment (in 1978) without job training, and α_i is the marginal effects of training on employment probability. I control for age (G_i), years of education (E_i), race (Black BLK_i and Hispanic HIS_i), and baseline employment status ($EMP75_i$).[17] I allow for an interaction term between the training treatment variable B_i and an indicator of whether an individual is below 24 years of age. A_i and α_i are individual-specific:

$$\begin{aligned}
\text{let } \Omega^{\text{emp78}}\left(\boldsymbol{x}_i\right) &= \exp\left(\theta_0 + \theta_1 G_i + \theta_2 G_i^2 + \theta_3 E_i + \theta_4 BLK_i + \theta_5 HIS_i + \theta_6 EMP75_i\right), \\
\text{then } A_i\left(\boldsymbol{x}_i\right) &= \frac{\Omega^{\text{emp78}}\left(\boldsymbol{x}_i\right)}{1+\Omega^{\text{emp78}}\left(\boldsymbol{x}_i\right)}, \\
\text{and } \alpha_i\left(\boldsymbol{x}_i\right) &= \frac{\Omega^{\text{emp78}}\left(\boldsymbol{x}_i\right)\cdot\exp\left(\theta_7 B_i + \theta_8 B_i \mathbf{1}\left\{G_i > 23\right\}\right)}{1+\Omega^{\text{emp78}}\left(\boldsymbol{x}_i\right)\cdot\exp\left(\theta_7 B_i + \theta_8 B_i \mathbf{1}\left\{G_i > 23\right\}\right)} - A_i\left(\boldsymbol{x}_i\right).
\end{aligned} \tag{20}$$

The allocation solutions accommodate structural or non-structural, parametric or non-parametric estimation and simulation results. Equation (20) provides an illustrative mapping. In this example, I am assuming that planner preferences aggregate over expected individual outcomes. As discussed earlier, the planner could also aggregate over possible realizations of individual outcomes with individual-specific uncertainty aggregations.

I plot the joint distribution of A_i (x-axis) and α_i (y-axis) in Figure 3.[18] The Utilitarian planner

17. Age ranges from 17 to 55 (median 23). Education ranges from 3 to 16 years (median 10). Hispanics account for 10 percent of the sample. Blacks account for 80 percent of the sample. $EMP75_i$ is equal to zero for those that reported zero wages in 1975 and one otherwise.

18. In practice, a program-manager might combine experimental estimates with scale-up observables $\boldsymbol{x}_i$ to

focuses on α_i and allocates in descending order of the y-axis. The Rawlsian planner focuses on A_i and allocates in ascending order of the x-axis. Here, the Atkinson-CES planner's problem considers α_i and A_i jointly. Conditional on A_i, α_i is higher for younger individuals. Overall, A_i and α_i are negatively correlated. This contrasts with the earlier $N = 2$ examples where the individual with lower A_i also has lower α_i. Despite the overall negative correlation, the possibility for trading off between needs and effectiveness arises due to the age interaction coefficient θ_8. From Figure 3, one could see that the individual with the highest value of α_i does not have the lowest value of A_i. Under the Utilitarian planner, the top-ranked individual on the allocation queue is a 23-year-old African American with nine years of education. Under the Rawlsian planner, the top-ranked individual is a 31-year-old African American with nine years of education.

A planner allocates training spots given the heterogeneous A_i and α_i. There are $\frac{722!}{(722-297)!\cdot 297!} \approx 7.4 \cdot 10^{210}$ possible ways of allocating 297 training spots among the 722 candidate recipients. Given Atkinson-CES preferences and Assumption 1,[19] I apply Theorem 1 to solve for optimal allocations as λ shifts from the Utilitarian to the Rawlsian boundary points. In Figure 4, I focus on eight individuals who are ranked at the 1st, 101st, 201st, 301st, 401st, 501st, 601st, and 701st positions along the optimal allocation queue under $\lambda = 0.99$. Three of these individuals have decreasing rankings (higher rank values) as λ moves towards -100. The ranking lines have different λ points where rankings shift away from the Utilitarian starting points.

I now consider the welfare differences between alternative allocation rules. The first alternative is to provide 297 training spots randomly using the observed NSW treatment assignments. The second alternative is to provide training spots to the 289 individuals with zero wages for 1975 and the 8 individuals with the lowest positive wages for 1975. Given expenditure minimization, if $\Delta^{\text{REV}} = 0.2$, the optimal allocation policy needs 20 percent fewer training spots to match the welfare under an alternative allocation rule. The results are shown in Figure 5. In this example, Δ^{REV} compared to the random allocation is equal to up to 0.25 for $\lambda \in (0, 1)$. As λ approaches the Rawlsian boundary, unless the individual who is the most in need (lowest A_i) receives a training spot, Δ^{REV} would approach 1. The Δ^{REV} losses from baseline wage conditioning (dashed blue line) are smaller, although still substantial compared to the optimal.

determine A_i and α_i among candidate recipients. Here, I use the observables from the experimental sample to illustrate ideas.

19. For binary problems, the non-increasing effects restriction of Assumption 1 does not apply.

Given estimates and available observables, the optimal allocation queue provides a scale to measure the relative welfare gains of alternative allocations. Depending on the expected outcomes of interest, each observable dimension impacts needs and effectiveness differently. The results here allow a program manager to run horse-races between alternative allocation policies given different planner preferences, allocation restrictions, and estimation-prediction assumptions.

6 Conclusion

Governments and aid agencies are often confronted with the challenge of allocating limited resources among a large pool of candidate recipients. When resources are discrete or when individual allocations face possibly binding constraints, the optimal allocation problem can become computationally intractable as the number of candidate recipients increases. Given the computational challenges, rather than finding the allocation results that maximize certain planning objectives, researchers might only be able to compare the effects of several policy alternatives or counterfactuals.

In this paper, I provide closed-form solutions and implementation algorithms for Atkinson-CES planning problems with discrete and bounded-continuous choice-sets. The solutions are characterized by optimal allocation queues, which are determined by planner preferences, individual allocation constraints, and heterogeneous outcomes across individuals with and without incremental allocations. Given non-increasing marginal returns of allocations, optimal allocation queues are invariant to resources.

The results could be applicable in a variety of policy-relevant settings with heterogeneous candidate recipients of allocations. Consider the problem that the World Food Program (WFP) might face in distributing nutritional supplements to reduce stunting among children. Given some observables at birth, the probability that children become stunted in the absence of nutritional aid could be heterogeneous. Prior research might also indicate that the marginal effects of nutritional aid on stunting probabilities are heterogeneous conditional on observables. Given these heterogeneities, the optimal allocation results in this paper could help WFP program managers determine which children should be prioritized to receive nutritional supplements to minimize stunting.

More broadly, the results of this paper can be useful in cost and benefit analysis of empirical allocation strategies. Policymakers potentially face a variety of allocation alternatives, including allocating universally and uniformly, allocating randomly, or allocating by sorting over a limited number of observables. The welfare gains—measured using the results of this paper in resource units as resource equivalent variations—from allocating optimally conditional on observables might be large or small depending on the distribution of observables, variations in policy constraints, and planner preferences. Beyond the optimal allocation problem, the closed-form solutions for discrete and bounded-continuous CES problems might also be useful in other contexts where CES functions are applied.

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Figure 1: Discrete Choice Resource Expansion Paths with $N = 2$

Figure 2: Bounded Continuous Resource Expansion Paths with $N = 2$

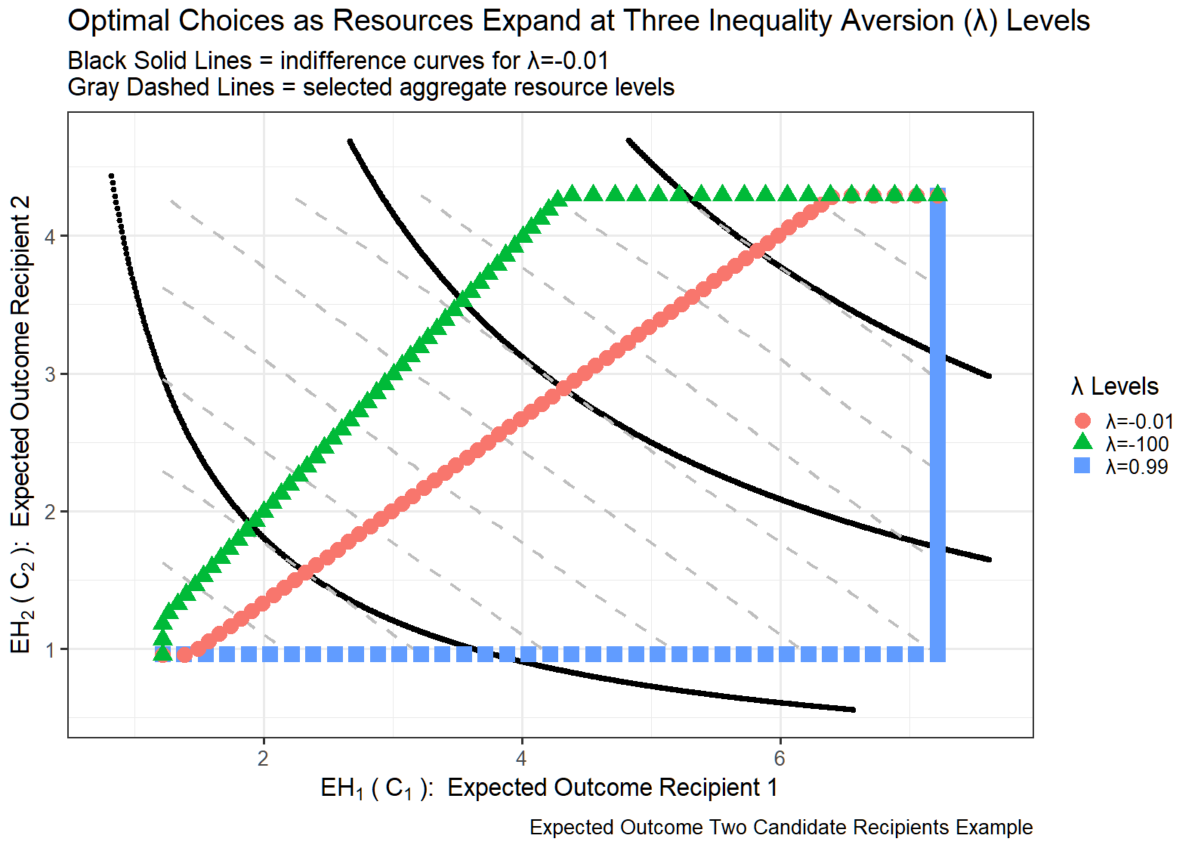
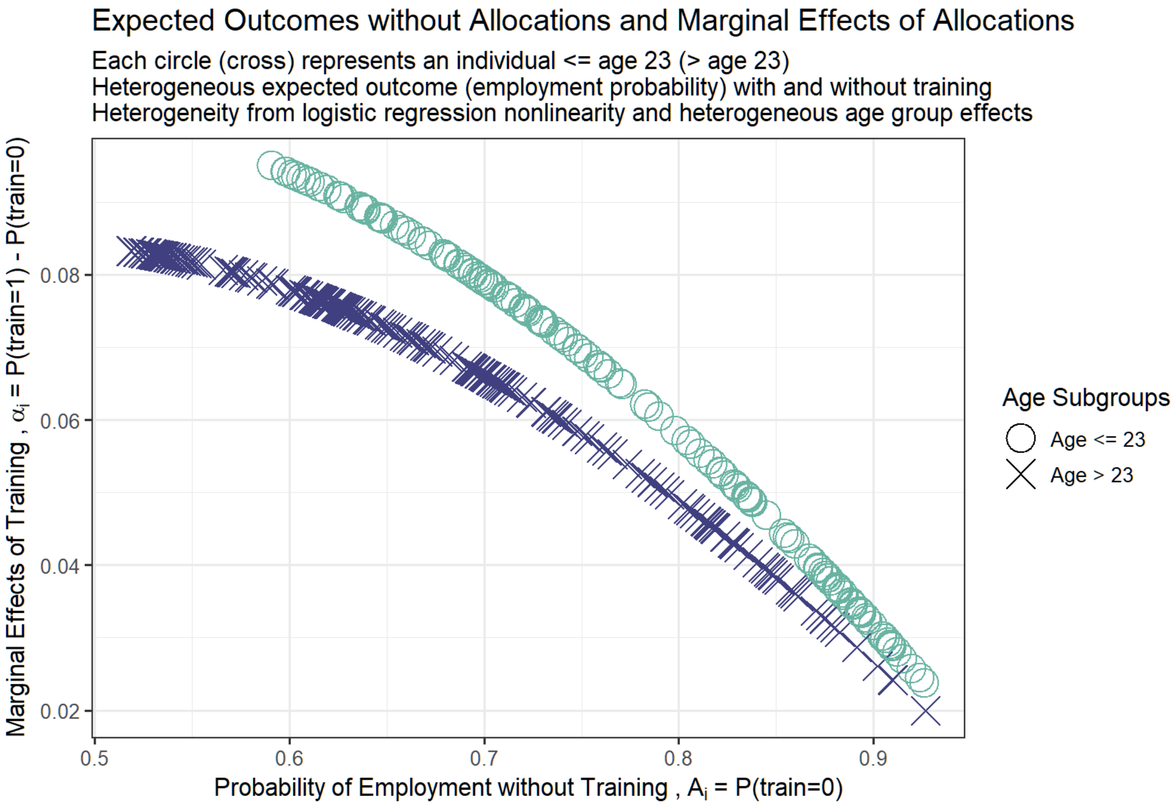


Figure 3: Logit–Employment Regression: Joint Distribution of α_i and A_i



Based on a logistic regression of the employment effects of a training RCT. Data: Lalonde (AER, 1986).

Figure 4: Logit–Employment Regression: Optimal Binary Allocation Queues

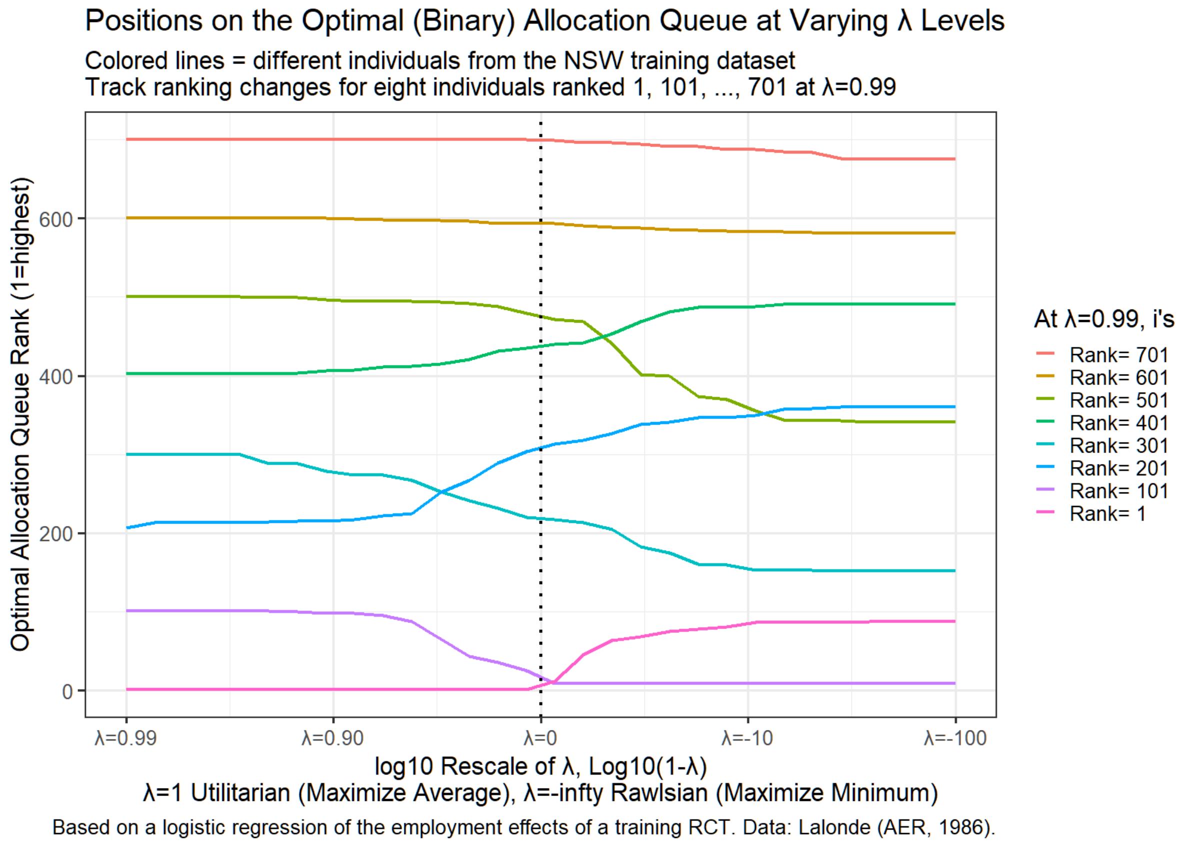
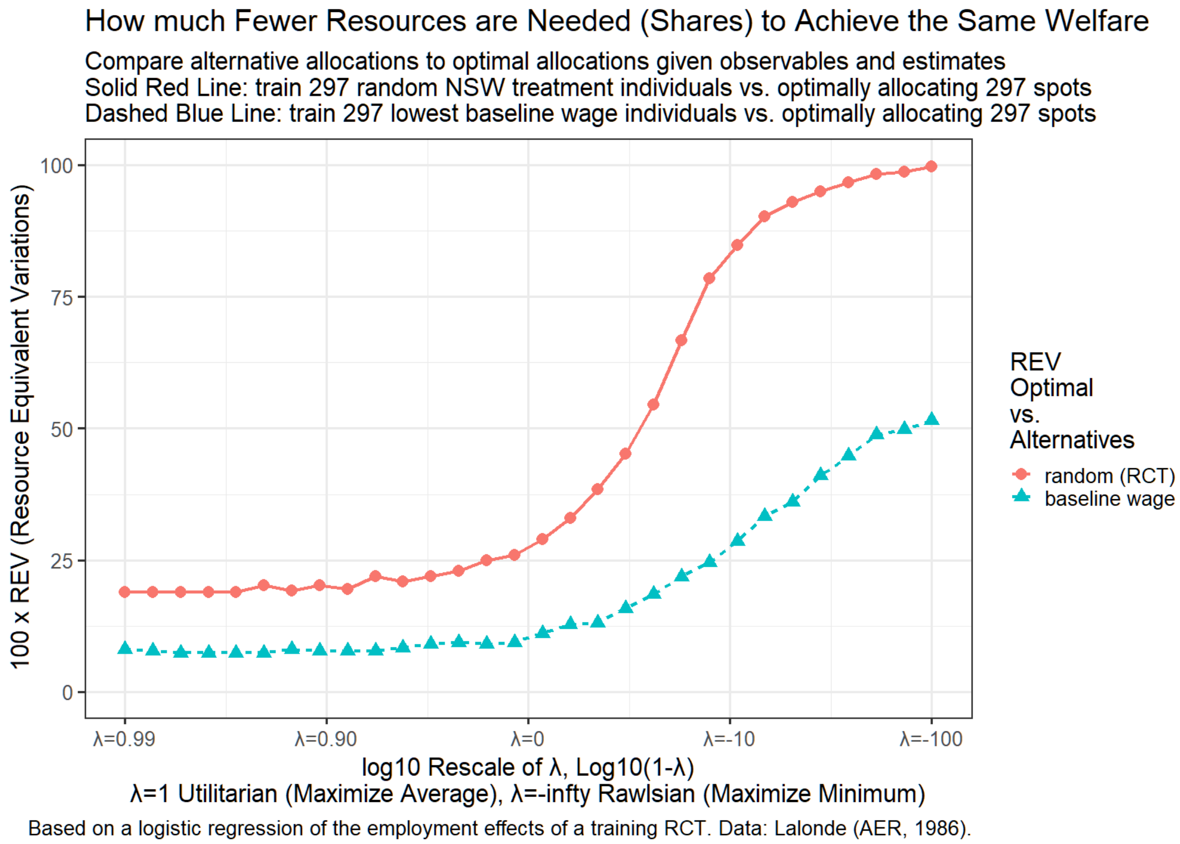


Figure 5: Logit–Employment: Resource Equivalent Variation Δ^{REV}



A Proofs

A.1 Discrete Allocation Proof

A.1.1 Binary Allocation

Theorem 1 applies to the binary allocation problem with $\underline{D}_i = \underline{D} = 0$ and $\bar{D}_i = \bar{D} = 1$. The choice set for a planner's binary allocation problem can be written as:

$$\mathcal{C}^B \equiv \left\{ \mathbf{B} = (B_1, \cdots, B_N) : B_i \in \{0,1\}, \text{ and, } \sum_{i=1}^{N} B_i \leq \widehat{W} \right\}. \tag{21}$$

Following Theorem 1, the optimal binary allocation function and allocation queue are:

$$\begin{gathered} B_i^*\left(\widehat{W}\right) = \mathbb{1}\left\{ \mathcal{Q}_i^B \leq \widehat{W} \right\} \\ \text{and } \mathcal{Q}_i^B = \sum_{j=1}^{N} \mathbb{1}\left\{ \frac{\beta_j}{\beta_i} \cdot \left(\frac{(A_j + \alpha_j)^\lambda - A_j^\lambda}{(A_i + \alpha_i)^\lambda - A_i^\lambda} \right) \geq 1 \right\}. \end{gathered} \tag{22}$$

Theorem 1 allows for ties: under binary allocations, if there are only two candidate recipients and they have identical A_i and α_i, both would have a queue rank of 2. For notational clarity, I will ignore the possibility of ties in the following proof.

I continue now to offer a proof for Theorem 1 when allocations are binary. In the following proof, the problem of allocating $\widehat{W}$ across N individuals where $\widehat{W} < N$ is reformulated as a problem of allocating 1 unit of allocation across individuals iteratively. The proof shows that the optimal allocation queue is based on a greedy-local comparison for the next available unit of allocation resource. The proof shows the simple result that with CES preference aggregation, the relative preference for allocating one unit of resource between two candidate recipients, who have yet to receive the binary allocation, is invariant to whether others have received allocations.

Proof of Theorem 1 (Binary Case): Given a vector of existing binary allocations $\widehat{\mathbf{B}}^{\mathbf{W}} = \left(\widehat{B}_1^W, \cdots, \widehat{B}_N^W\right)$, where $\widehat{B}_i^W \in \{0,1\}$ and $\sum_{i=1}^{N} \widehat{B}_i^W = W$, consider the problem of maximizing the Atkinson-CES planner's objective function on the constraint set $\mathcal{C}^{\widehat{B}}$

$$\mathcal{C}^{\widehat{B}}\left(\widehat{\mathbf{B}}^{\mathbf{W}}\right) \equiv \left\{ \mathbf{B} = (B_1, \cdots, B_N) : B_i \in \{0,1\}, B_i + \widehat{B}_i^W \leq 1, \text{ and, } \sum_{i=1}^{N} B_i \leq 1 \right\}, \tag{23}$$

where the planner allocates one unit of resource given existing allocations $\widehat{\mathbf{B}}^{\mathbf{W}}$. While the choice set $\mathcal{C}^B$ contains N choose $\widehat{W}$ combinations of elements, $\mathcal{C}^{\widehat{B}}$ only has at most N elements.

Given $\mathcal{C}^{\widehat{B}}$, the planner would optimally allocate the one unit of resource to individual i, someone who has not previously received the binary allocation (i.e. $\widehat{B}_i^W = 0$), if and only if,

$$\begin{aligned} &\left(\beta_i \left(A_i + \alpha_i\right)^\lambda + \textstyle\sum_{\tilde{i}=1}^N \beta_{\tilde{i}} \left(A_{\tilde{i}} + \alpha_{\tilde{i}} \widehat{B}_{\tilde{i}}^W\right)^\lambda - \beta_i A_i^\lambda\right)^{\frac{1}{\lambda}} \\ \geq &\left(\beta_j \left(A_j + \alpha_j\right)^\lambda + \textstyle\sum_{\tilde{i}=1}^N \beta_{\tilde{i}} \left(A_{\tilde{i}} + \alpha_{\tilde{i}} \widehat{B}_{\tilde{i}}^W\right)^\lambda - \beta_j A_j^\lambda\right)^{\frac{1}{\lambda}} \end{aligned} \quad \forall j \text{ where } \widehat{B}_j^W = 0 \, . \tag{24}$$

with $\lambda \in (-\infty, 1]$, the condition in Equation (24) simplifies to

$$\begin{aligned} &\left(\beta_i \left(A_i + \alpha_i\right)^\lambda - \beta_i A_i^\lambda\right) \geq \left(\beta_j \left(A_j + \alpha_j\right)^\lambda - \beta_j A_j^\lambda\right) \quad \text{if } \ 0 < \lambda \leq 1 \\ \text{and } \ &\left(\beta_i \left(A_i + \alpha_i\right)^\lambda - \beta_i A_i^\lambda\right) \leq \left(\beta_j \left(A_j + \alpha_j\right)^\lambda - \beta_j A_j^\lambda\right) \quad \text{if } \ \lambda \leq 0 \, . \end{aligned} \tag{25}$$

Given Assumption 1, $A_i + \alpha_i > A_i > 0$.[20] Both sides of the inequalities above are positive when $0 < \lambda \leq 1$ and non-positive when $\lambda \leq 0$. Hence, for i where $\widehat{B}_i^W = 0$,

$$\widehat{B}_i^* \left(\widehat{\mathbf{B}}^{\mathbf{W}}\right) = 1 \text{ iff } \frac{\beta_i}{\beta_j} \left(\frac{\left(A_i + \alpha_i\right)^\lambda - A_i^\lambda}{\left(A_j + \alpha_j\right)^\lambda - A_j^\lambda}\right) \geq 1 \ \forall j \text{ with } \widehat{B}_j^W = 0. \tag{26}$$

With $\widehat{B}_i^W = \widehat{B}_j^W = 0$, unless $\lambda = 1$, the marginal welfare gains from allocating to i and j are different in iterations W and $W - 1$. However, the planner's relative preferences between i and j in Equation (26) is not a function of $\widehat{\mathbf{B}}^{\mathbf{W}}$: given that preferences are homothetic over expected outcomes, relative preferences over individuals who have yet to receive allocations are invariant to prior allocations. Given this, Equation (26) can be rewritten as:

$$B_i^* \left(\widehat{\mathbf{B}}^{\mathbf{W}}\right) = 1 \text{ iff } \sum_{j=1}^N \mathbb{1} \left\{ \frac{\beta_j}{\beta_i} \left(\frac{\left(A_j + \alpha_j\right)^\lambda - A_j^\lambda}{\left(A_i + \alpha_i\right)^\lambda - A_i^\lambda}\right) \geq 1 \right\} = W + 1 \, . \tag{27}$$

Given $\widehat{\mathbf{B}}^{\mathbf{W=0}}$, $B_i^* \left(\widehat{\mathbf{B}}^{\mathbf{W=0}}\right)$ allocates the first unit of resource. $\widehat{B}_i^{W=1} = \widehat{B}_i^{W=0} + B_i^* \left(\widehat{\mathbf{B}}^{\mathbf{W=0}}\right)$, and $B_i^* \left(\widehat{\mathbf{B}}^{\mathbf{W=1}}\right)$ allocates the second unit. Given resource $\widehat{W}$, $B_i^* \left(\widehat{\mathbf{B}}^{\mathbf{W=\widehat{W}-1}}\right)$ allocates the final unit. Hence, each element of the optimal allocation queue is

$$\mathcal{Q}_i^B = \min \left\{ W : W \in \{1, ..., N\}, \text{and, } \widehat{B}_i^W = 1 \right\} \, , \tag{28}$$

which is equal to the $\widehat{W}$ required for i to optimally receive the binary allocation. Equation (28) is equivalent to the summation in Equation (22). ■

A.1.2 Discrete Allocation

Without restrictions on α_{il}, the discrete targeting queue might shift with $\widehat{W}$. Suppose $N = 2$, $\bar{D} = 2$, and $(\alpha_{1,1} = 1, \alpha_{1,2} = 0, \alpha_{2,1} = 0, \alpha_{2,2} = \infty)$. If $\widehat{W} = 1$, a Utilitarian planner allocates

20. The third restriction of Assumption 1 means $A_i > 0$ under the binary allocation problem.

to $i = 1$. If $\widehat{W} = 2$, the Utilitarian allocates to $i = 2$ only. The resource expansion path would bend backward, which prevents the derivation of a closed-form and resource-invariant optimal allocation queue. I now offer a proof of Theorem 1 given the non-increasing effects assumption of Theorem 1. For the binary allocation problem, the non-increasing effects restriction was irrelevant because there is only one possible allocation increment. Similar to the binary proof, for notational clarity, I ignore ties.

Following on the binary proof, the discrete allocation proof shows that the optimal allocation queue of Theorem 1 is based on sequential local comparisons over the next unit of resource available. Essentially, under Atkinson-CES preference aggregation and non-increasing marginal gains for incremental allocations, a sequential greedy-local solution is not only globally optimal but also invariant to aggregate resources.

Proof of Theorem 1: Under Assumption 1, α weakly decreases, $\alpha_{i,l-1} \geq \alpha_{i,l}\, \forall\, l, i$. Given this, optimality is satisfied by iteratively considering the Atkinson planner's problem for the next available unit of resource.

I first describe the iterative procedure, which is similar to the procedure in the binary proof. Let $\widehat{\mathbf{D}}^{\mathbf{W}} = \left(\widehat{D}_1^W, \cdots, \widehat{D}_N^W\right)$: $\widehat{D}_i^W \in \{0, 1, \cdots, \bar{D}_i\}$ and $\sum_{i=1}^N \widehat{D}_i^W = W$. Consider the problem of maximizing the planner's objective function on the constraint set $\mathcal{C}^{\widehat{D}}$

$$\mathcal{C}^{\widehat{D}}\left(\widehat{\mathbf{D}}^{\mathbf{W}}\right) \equiv \left\{ \boldsymbol{D} = (D_1, \cdot\cdot, D_N) : D_i \in \{0, 1\}, D_i + \widehat{D}_i^W \leq \bar{D}_i, \text{ and, } \sum_{i=1}^N D_i \leq 1 \right\}. \quad (29)$$

where the planner allocates one unit of resource given existing allocations $\widehat{\mathbf{D}}^{\mathbf{W}}$.

Given $\mathcal{C}^{\widehat{D}}$, i with $\widehat{D}_i^W < \bar{D}_i$ optimally receives allocation, if and only if, $\forall j$ where $\widehat{D}_j^W < \bar{D}_j$,

$$\begin{aligned}
&\left(\beta_i \left(A_i + \sum_{l=1}^{\widehat{D}_i^W+1} \alpha_{il} \right)^{\lambda} + \sum_{\tilde{i}=1}^N \beta_{\tilde{i}} \left(A_{\tilde{i}} + \sum_{l=1}^{\widehat{D}_{\tilde{i}}^W} \alpha_{\tilde{i}l} \right)^{\lambda} - \beta_i \left(A_i + \sum_{l=1}^{\widehat{D}_i^W} \alpha_{il} \right)^{\lambda} \right)^{\frac{1}{\lambda}} \\
\geq &\left(\beta_j \left(A_j + \sum_{l=1}^{\widehat{D}_j^W+1} \alpha_{jl} \right)^{\lambda} + \sum_{\tilde{i}=1}^N \beta_{\tilde{i}} \left(A_{\tilde{i}} + \sum_{l=1}^{\widehat{D}_{\tilde{i}}^W} \alpha_{\tilde{i}l} \right)^{\lambda} - \beta_j \left(A_j + \sum_{l=1}^{\widehat{D}_j^W} \alpha_{jl} \right)^{\lambda} \right)^{\frac{1}{\lambda}}
\end{aligned} . \quad (30)$$

Following the Proof for the binary case and given Equation (30), for i, if $\widehat{D}_i^W < \bar{D}_i$,

$$D_i^*\left(\widehat{\mathbf{D}}^{\mathbf{W}}\right) = 1 \text{ iff } \sum_{j=1}^{N}\sum_{\widetilde{l}=1}^{\bar{D}_j} \mathbb{1}\left\{\frac{\beta_j}{\beta_i}\left(\frac{\left(A_j + \sum_{l=1}^{\widetilde{l}} \alpha_{jl}\right)^{\lambda} - \left(A_j + \sum_{l=1}^{\widetilde{l}-1} \alpha_{jl}\right)^{\lambda}}{\left(A_i + \sum_{l=1}^{\widehat{D}_i^W+1} \alpha_{il}\right)^{\lambda} - \left(A_i + \sum_{l=1}^{\widehat{D}_i^W} \alpha_{il}\right)^{\lambda}}\right) \geq 1\right\} = W+1\,. \tag{31}$$

At the W^{th} iteration, Equation (31) shows that even though the same unallocated increment had different marginal effects on welfare as W rose, the relative preferences over the next unit of unallocated increments are invariant as allocations are provisioned.

$\widehat{D}_i^{W+1} = \widehat{D}_i^W + D_i^*\left(\widehat{\mathbf{D}}^{\mathbf{W}}\right)$. Each element of the optimal allocation queue is

$$Q_{il}^D = \min\left\{W : W \in \left\{1, ..., \sum_{i=1}^{N} \bar{D}_i\right\}, \text{and, } \widehat{D}_i^W = l\right\}\,. \tag{32}$$

which is equal to the $\widehat{W}$ level of aggregate resources needed to optimally allocate to the l^{th} increment of individual i. Equation (32) is equivalent to the summation in Equation (8).

The iterative optimal solution Q_{il}^D is only optimal if the resource expansion path does not bend backward. This means that if it is optimal to allocate consecutively to i, it is also optimal to allocate to i over $\mathcal{C}^{\widehat{D}}$ iteratively. Without loss of generality, suppose $W = 0$, the requirement is, $\forall j$,

$$\text{if}\left(\frac{(A_i+\alpha_{i,1}+\alpha_{i,2})^{\lambda} - (A_i+\alpha_{i,1})^{\lambda}}{(A_j+\alpha_{j,1})^{\lambda} - (A_j)^{\lambda}}\right) \geq \frac{\beta_j}{\beta_i}, \text{then}\left(\frac{(A_i+\alpha_{i,1})^{\lambda} - (A_i)^{\lambda}}{(A_j+\alpha_{j,1})^{\lambda} - (A_j)^{\lambda}}\right) \geq \frac{\beta_j}{\beta_i}. \tag{33}$$

Assumption $\alpha_{i,l-1} \geq \alpha_{i,l}$ satisfies Equation (33). For the inequalities in Equation (33), comparisons are based on differences in the numerators. Let $g(A\,;\,\alpha) = (A+\alpha)^{\lambda} - (A)^{\lambda}$. Given $\alpha_{i1} \geq \alpha_{i2}$, if $0 < \lambda < 1$, then $0 < g(A_i+\alpha_{i1}\,;\,\alpha_{i2}) \leq g(A_i+\alpha_{i1}\,;\,\alpha_{i1}) < g(A_i\,;\,\alpha_{i1})$. This satisfies in the condition in Equation (33). If $\lambda < 0$, both the numerator and the denominator are negative. We have, given $\alpha_{i1} \geq \alpha_{i2}$, $0 > g(A_i+\alpha_{i1}\,;\,\alpha_{i2}) \geq g(A_i+\alpha_{i1}\,;\,\alpha_{i1}) > g(A_i\,;\,\alpha_{i1})$. This also satisfies the condition in Equation (33). ■

A.2 Continuous Allocation Proof

In contrast to the results from Theorem 1 for discrete allocation problems, results from Theorem 2 allow for individual-specific prices. Due to indivisibility, a discrete resource expansion path might bend backward when the costs of acquiring individual-specific allocations differ: allocations with high marginal effects but also high unit costs might displace lower unit costs allocations as aggregate resources expand. When the resource expansion path is backward-bending, the optimal allocation queue loses its invariance with respect to aggregate resources.

A.2.1 Lower Bound

Theorem 2 applies to constant-returns lower-bounded continuous allocation problems where $\bar{C}_i = \bar{C} = \infty$. Given that lower-bounds can be subsumed under A_i without loss of generality, the choice set for lower-bounded continuous allocation problems can be stated as:

$$\mathcal{C}^L \equiv \left\{ \mathbf{L} = (L_1, \cdots, L_N) : 0 \leq L_i, \text{ and, } \sum_{i=1}^{N} P_i \cdot L_i \leq \widehat{W} \right\}. \tag{34}$$

To make the effects of prices more transparent, Equation (34) incorporates individual-specific input prices P_i.

Lower-bounded start-queue $\underline{\mathbf{Q}}^L$ and start-knots $\underline{\mathbf{K}}^L$ are equivalent to $\underline{\mathbf{Q}}^C$ and $\underline{\mathbf{K}}^C$ from Theorem 2. The lower-bounded problems do not have end-queues or end-knots. Following Theorem 2, the optimal constant-returns lower-bounded continuous allocation functions are:

$$L_i^*\left(\widehat{W}\right) = \max\left\{0,\ \phi^{\mathrm{y}}_{i,\mathbb{I}} + \phi^{\mathrm{s}}_{i,\mathbb{I}} \cdot L_{\mathbb{I}}\left(\widehat{W}\right)\right\}, \tag{35}$$

$$\text{and } L_{\mathbb{I}}\left(\widehat{W}\right) = \frac{\widehat{W} - \sum_{j=1}^{N}\left(P_j \cdot \phi^{\mathrm{y}}_{j,\mathbb{I}}\right) \cdot \mathbb{1}\left\{\underline{\mathcal{K}}^L_j \leq \widehat{W}\right\}}{\sum_{j=1}^{N}\left(P_j \cdot \phi^{\mathrm{s}}_{j,\mathbb{I}}\right) \cdot \mathbb{1}\left\{\underline{\mathcal{K}}^L_j \leq \widehat{W}\right\}}. \tag{36}$$

The linear spline in Equation (36) has N knots.

I continue now to offer a proof for Theorem 2 when allocations are lower-bounded.

Proof of Theorem 2 (Lower-Bounded Case): Given Assumption 2, First Order Conditions from the Atkinson planner's problem give rise to the unconstrained optimal relative allocation rule between individuals i and j:

$$\begin{aligned}
\left(A_j + \alpha_j L_j\right) &= \left(\frac{\beta_j \alpha_j P_j^{-1}}{\beta_i \alpha_i P_i^{-1}}\right)^{\frac{1}{1-\lambda}} A_i + \left(\frac{\beta_j \alpha_j P_j^{-1}}{\beta_i \alpha_i P_i^{-1}}\right)^{\frac{1}{1-\lambda}} \alpha_i L_i \\
L_j &= \left(\left(\frac{\beta_j \alpha_j P_j^{-1}}{\beta_i \alpha_i P_i^{-1}}\right)^{\frac{1}{1-\lambda}} \frac{A_i}{\alpha_j} - \frac{A_j}{\alpha_j}\right) + \left(\frac{\beta_j \alpha_j P_j^{-1}}{\beta_i \alpha_i P_i^{-1}}\right)^{\frac{1}{1-\lambda}} \frac{\alpha_i}{\alpha_j} \cdot L_i \\
L_j^{rela}\left(L_i\right) &= \phi^{\mathrm{y}}_{j,i} + \phi^{\mathrm{s}}_{j,i} \cdot L_i
\end{aligned} \tag{37}$$

Y-intercept $\phi^{\mathrm{y}}_{j,i}$ indicates if optimal relative allocation for j is positive when the allocation for i is zero. There exists an individual $i = \mathbb{I}$, relative to whom $\phi^{\mathrm{y}}_{j,\mathbb{I}} \leq 0$ for all j.

Given Equation (37) and $\sum_{j=1}^{N} P_j \cdot L_j = \widehat{W}$, the unconstrained optimal choice $L_i^{*,\text{unc}}$ is

$$L_i^{*,\text{unc}} = \frac{\widehat{W} - \sum_j\left(P_j \cdot \phi^{\mathrm{y}}_{j,i}\right)}{\sum_j\left(P_j \cdot \phi^{\mathrm{s}}_{j,i}\right)}. \tag{38}$$

$L_i^{*,\text{unc}}$ would be optimal if the planner could allocate $\widehat{W}$ as well as reallocate A_i. But the problem has N positivity constraints. Complementary slackness requires that either relative optimality (Equation (37)) is satisfied or individual constraints bind. Hence, to arrive at the constrained optimal choice, in Equation (39), I sum $L_j^{rela}\left(L_i\right)$ across all j, but only when L_i exceeds the j-specific relative x-intercepts $\frac{-\phi_{j,i}^{\text{y}}}{\phi_{j,i}^{\text{s}}}$:

$$\widehat{W}=\sum_j P_j\cdot\phi_{j,i}^{\text{y}}\cdot\mathbb{1}\left\{\frac{-\phi_{j,i}^{\text{y}}}{\phi_{j,i}^{\text{s}}}\leq L_i\right\}+\left(\sum_j P_j\cdot\phi_{j,i}^{\text{s}}\cdot\mathbb{1}\left\{\frac{-\phi_{j,i}^{\text{y}}}{\phi_{j,i}^{\text{s}}}\leq L_i\right\}\right)\cdot L_i \tag{39}$$

The y-intercept of Equation (39) that is aggregated relative to $L_{i=\mathbb{I}}$ is zero. Hence, individual $i=\mathbb{I}$ is the first (top) ranked candidate recipient. Positions along the start-queue $\underline{\mathbf{Q}}^L$ are ordered by x-intercepts relative to $L_{i=\mathbb{I}}$, $\frac{-\phi_{j,\mathbb{I}}^{\text{y}}}{\phi_{j,\mathbb{I}}^{\text{s}}}$. Given its functional form, the ascending order of x-intercepts, with least element $i=\mathbb{I}$, is determined by $A_i/\left(\alpha_i\beta_i P_i^{-1}\right)^{\frac{1}{1-\lambda}}$.

With strictly positive marginal effects (Assumption 2), inverting Equation (39) relative to $L_{i=\mathbb{I}}$ generates the linear spline $L_{\mathbb{I}}\left(\widehat{W}\right)$,

$$L_{\mathbb{I}}\left(\widehat{W}\right)=\left(\frac{\widehat{W}-\sum_{j=1}^{N}\left(P_j\cdot\phi_{j,\mathbb{I}}^{\text{y}}\right)\cdot\mathbb{1}\left\{\underline{\mathcal{K}}_j^L\leq\widehat{W}\right\}}{\sum_{j=1}^{N}\left(P_j\cdot\phi_{j,\mathbb{I}}^{\text{s}}\right)\cdot\mathbb{1}\left\{\underline{\mathcal{K}}_j^L\leq\widehat{W}\right\}}\right), \tag{40}$$

where knots $\underline{\mathbf{K}}^L$ of $L_{\mathbb{I}}\left(\widehat{W}\right)$ are found by evaluating Equation (39) at each x-intercept relative to $L_{\mathbb{I}}$:

$$\underline{\mathcal{K}}_j^L=\sum_{i=1}^{N}\left(P_i\cdot\phi_{i,\mathbb{I}}^{\text{y}}+P_i\cdot\phi_{i,\mathbb{I}}^{\text{s}}\cdot\frac{-\phi_{j,\mathbb{I}}^{\text{y}}}{\phi_{j,\mathbb{I}}^{\text{s}}}\right)\cdot\mathbb{1}\left\{\underline{\mathcal{Q}}_i^L\leq\underline{\mathcal{Q}}_j^L\right\} \tag{41}$$

Combining $L_{\mathbb{I}}\left(\widehat{W}\right)$ and the relative optimality conditions, optimal allocations for all candidate recipients are equal to: $L_i^*\left(\widehat{W}\right)=\max\left\{0,\ \phi_{i,\mathbb{I}}^{\text{y}}+\phi_{i,\mathbb{I}}^{\text{s}}\cdot L_{\mathbb{I}}\left(\widehat{W}\right)\right\}$. ■

A.2.2 Lower and Upper Bounds

Results for the constant-returns upper- and lower-bounded continuous allocation problem relies on the same logic as the proof for constant-returns lower-bounded continuous allocation problems. Given their similarities, I do not provide a separate proof here. Overall, the addition of upper bounds introduces N additional constraints, which become additional knots in the constant-returns allocation spline relative to $i=\mathbb{I}$.

The relative optimality Equation (37) remains the same. For the doubly bounded problem, Equation (39) is changed to sum $C_j^{rela}\left(C_i\right)$ across all j when C_i is both above the j-specific relative x-intercepts $\frac{-\phi_{j,i}^{\text{y}}}{\phi_{j,i}^{\text{s}}}$ and below $\frac{\bar{C}_j-\phi_{j,i}^{\text{y}}}{\phi_{j,i}^{\text{s}}}$. When C_i exceeds a j specific upper threshold, $\bar{C}_j$ is included in the summation.

Equation (40) also remains similar, but now $\widehat{W}$ is splined by lower bounds $\underline{K}_j^C$ and upper bounds $\bar{K}_j^C$. Additionally, the within-segment aggregate inverted y-intercept includes the aggregate costs of allocating $\bar{C}_i$ to individuals who have reached their maximum allocations in this splined-segment. These are shown in Equation (15) of Theorem 2.

B Implementation Examples

A variety of examples and implementation tutorials are shown on the paper's optimal allocation website. All theorems are programed as functions of the R package PrjOptiAlloc.